Perturbations in Two- and Three-Dimensional Transonic Flows

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The difficulty of treating the perturbation of transonic flow, during which shock waves change position, can be overcome by using a distorted coordinate system in which the locations of all shock waves do not change; the distortion is found as part of the solution. This device leads to a relation that allows a range of flows, with differing shock locations, to be related algebraically to two known "calibration" flows. Results for flows around finite wings, including those with multiple, intersecting shock waves, are presented. A typical computing time for such examples is 0.3 s on a CDC 7600 computer.

Nomenclature

C_D	= drag coefficient
C_L	= lift coefficient
C_m	= pitching moment coefficient about the leading edge
$C_p(x,y,z)$	= pressure coefficient
k	= transonic parameter
M	= Mach number
q	= exponent in k
s	= semispan
(x,y,z)	= scaled Cartesian coordinate system
$(\bar{x},\bar{y},\bar{z})$	= physical Cartesian coordinate system
(x',y',z')	= strained coordinate system
$x_1(x',y')$	= straining function of x
$x_L(y')$	= location of leading edge
$x_s(y')$	= shock location
$x_T(y')$	= location of trailing edge
$y_1(y')$	= straining function for y
y_T	= shock termination point in y direction
$\bar{z}_L(\bar{x},\bar{y})$	= geometry of lower surface of the wing
z_T	= shock termination point in z direction
$\bar{z}_u(\bar{x},\bar{y})$	= geometry of upper surface of the wing
$Z_L(x,y)$	$= k/\beta^3 \bar{z}_L(\bar{x},\bar{y})$
$Z_u(x,y)$	$= k/\beta^3 \bar{z}_u(\bar{x},\bar{y})$
$Z_{Lp}(x,y)$	= measure of perturbation of lower surface of wing
$Z_{up}(x,y)$	= measure of perturbation of upper surface of wing
α	= angle of attack
β	$= (1-M_\infty^2)^{1/2}$
γ	= ratio of specific heats
$\delta x_s(y')$	= increment in $x_s(y')$
δy_T	= increment in y_T
δz_T	= increment in z_T
Δ	= perturbation parameter
ϵ	= perturbation parameter
θ	= angle of twist
$\bar{\phi}(\bar{x},\bar{y},\bar{z})$	= physical perturbation potential
$\phi(x,y,z)$	$= k/\beta^2 \bar{\phi}(\bar{x},\bar{y},\bar{z})$

Subscripts

0	= base solution
1	= calibration solution
∞	= freestream value

Superscripts

(0)	= base solution
(1)	= calibration solution
Δ	= perturbation parameter Δ
ϵ	= perturbation parameter ϵ

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Index category: Transonic Flow.
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I. Introduction

IN inviscid flows described by a linear equation and with linear boundary conditions, an unknown flow frequently can be obtained from a known flow by the application of a scaling factor. For example, the Prandtl-Glauert formula relates flows at one freestream Mach number to flows at another by a simple scaling. Also, a complex flow may be obtained from a range of simple known flows by using the principle of superposition. An example for incompressible flow is the use of a combination of elementary source solutions to calculate the flows around an airfoil. In problems for which a simple scaling relates a range of flows with differing values of a flow parameter, such as angle of attack, freestream Mach number, etc., only one "base" solution need be known in order to compute the range of flows. Because of the linearity of the problem, this type of solution is generally valid for all values of the flow parameter undergoing change, although it is probable that only a range of parameter variation will have any physical meaning.

In continuous flows described by a nonlinear equation and/or nonlinear boundary conditions, the nonlinear problem can be perturbed about some known base solution that leads to a linear equation for some perturbation variable. The behavior of the flow with the variation of some flow parameter can be found if another solution, the "calibration" solution, is known for some value of the parameter other than that characterizing the base solution. The range of validity of this type of solution is limited by the range of variation of the flow parameter compatible with the linearization process, that is, when the neglected terms in the perturbation analysis are still, in fact, negligible. This type of solution can be obtained if the flow variables in question, for example, pressure coefficient, vary continuously with the perturbation parameter.

If shock waves are present in the flow, then the condition that the flow varies continuously with the perturbation parameter does not hold if the shock waves move during the perturbation, since the flow variables can change discontinuously in the region bounded by the extremities of the shock motion. It is shown in Ref. 1 that, for two-dimensional nonlifting flows, this problem can be treated easily if one uses a strained coordinate system in which the location of the shock waves remains constant throughout the perturbation. The choice of the strained coordinate system is fairly arbitrary. The technique is restricted to perturbations during which no loss or generation of shock waves occurs. In the method of Ref. 1, the perturbation variables are linear functions of the perturbation parameters, although the overall solution is nonlinear; the nonlinearity is introduced implicitly through the use of the strained coordinate system. The solutions computed by this method require two known solutions, namely, a base solution and a calibration solution.

In the present paper, the fundamental concept of Ref. 1 is extended to two-dimensional lifting flows and to three-dimensional lifting flows with multiple, intersecting shock waves. Examples of the computation of such flows are presented.

Apart from the computation of flow variables such as the pressure coefficient, other flow characteristics such as the lift, pitching moment, and drag coefficients can be calculated from the base and calibration solutions. In this paper, the drag coefficient variation with freestream Mach number (the drag rise curve) is calculated for a 10% biconvex airfoil. Also, the variation of the lift and pitching moment coefficient with angle of attack for an NACA 64A410 airfoil is found. Within the confines of the perturbation theory, the algebraic formulas resulting from the present method indicate analytically the dependence of the lift, moment, and drag coefficients on the perturbation parameters. Because of the linearity of the equation governing the perturbation flow variables, the effect on the flow of more than one perturbation parameter can be investigated by using superposition. In this way, a range of parameters may be used, provided that the basic restriction of no loss or generation of shock waves throughout the perturbation is satisfied. In general, an investigation of an N parameter problem requires $N+1$ known solutions, that is, one base solution and N calibration solutions. An example of such a constructed flow around a three-dimensional wing for changes in angle of attack and twist is presented.

As mentioned earlier, the choice of the strained coordinate system is fairly arbitrary, and it is important to establish whether this choice can affect the overall result. The effect of changing the form of the straining in the coordinate system is discussed in the Appendix, and an example is computed by using two different coordinate systems. The results of both calculations agree to within the limits of numerical accuracy. The computing times for the perturbation problems are very small, being about 0.3 s on a CDC 7600 for the three-dimensional intersecting shock example.

II. Fundamental Concept

The basic problem is to obtain from two or more solutions an algebraic relation that connects the flow variables for a range of one or more flow parameters, thus leading to rapid computation of these related flows. The main difficulty, as noted in Sec. I, is in the treatment of the moving shock waves. The fundamental concept of the treatment of the moving discontinuities in transonic flow perturbations is given in Ref. 1, which also contains a detailed analysis for nonlifting flows with a single shock wave.

Briefly, the idea is that the problem is reformulated in a strained coordinate system in which the shock waves remain at the same location throughout the perturbation, and hence the difficulties associated with the moving shocks do not arise explicitly. The required straining is then found as part of the solution. The basic equations in this strained coordinate system are then perturbed about some known flow to give a linear equation for the perturbation quantities. Once these perturbation quantities are known, the total perturbed solution in the real physical coordinates is then obtained. The major restriction of the theory is that shock waves may not be lost or generated during the perturbation. Because of difficulties in presenting a unified detailed theory for the two- and three-dimensional cases, each case is treated separately.

III. Two-Dimensional Flows

The most common characteristic of a two-dimensional discontinuous transonic flow is that there is a single shock wave on at least one of the surfaces of the airfoil; the lower surface may be shock-free. It is to this type of flow that the following detailed analysis is applied, although the basic principles may be applied to more complex two-dimensional flows.

A. Basic Equations

The basic equation to be considered here is a scaled form of the transonic small-disturbance equation namely,

$$\phi_{xx}+\phi_{zz}=\phi_x\phi_{xx} \tag{1}$$

where (x,z) is a Cartesian coordinate system, with x aligned with the airfoil chord and related to the physical coordinate system $(\bar{x},\bar{z})$ by the transformation

$$x=\bar{x}, \quad z=\beta\bar{z} \tag{2}$$

where, if M_∞ is the freestream Mach number, then

$$\beta=(1-M_\infty^2)^{1/2}$$

The scaled velocity potential $\phi(x,z)$ is related to the perturbation velocity potential $\bar{\phi}(\bar{x},\bar{z})$ by the transformation

$$\phi(x,z)=(k/\beta^2)\bar{\phi}(\bar{x},\bar{z}) \tag{3}$$

where k is a transonic parameter of the form

$$k=(\gamma+1)M_\infty^q \tag{4}$$

γ is the ratio of specific heats, and q is an exponent that is arbitrary within the confines of small disturbance theory.

The pressure coefficient $C_p(\bar{x},\bar{z})$ is defined by

$$C_p(\bar{x},\bar{z})=-2\bar{\phi}_{\bar{x}}(\bar{x},\bar{z})=-2(\beta^2/k)\phi_x(x,z) \tag{5}$$

In the transformed variables of Eqs. (2) and (3), the appropriate thin airfoil boundary conditions for a perturbation of Eq. (1) are

$$\phi_z(x+0)=Z_{u_x}(x)+\epsilon Z_{up_x}(x) \tag{6a}$$

$$\phi_z(x,-0)=Z_{L_x}(x)+\epsilon Z_{Lp_x}(x) \tag{6b}$$

where, if $\bar{z}=\bar{z}_u(\bar{x})$ and $\bar{z}=\bar{z}_L(\bar{x})$ denote the geometry of the upper and lower surfaces of the airfoil, respectively, then

$$Z_u(x)=(k/\beta^3)\bar{z}_u(\bar{x}) \tag{7a}$$

$$Z_L(x)=(k/\beta^3)\bar{z}_L(\bar{x}) \tag{7b}$$

and $Z_{up}(x)$, $Z_{Lp}(x)$ denote the form of the perturbation to the airfoil on the upper and lower surfaces, respectively; ϵ is a small parameter denoting the magnitude of the perturbation.

As mentioned in Sec. II, the idea is to perturb Eq. (1) about some known solution, ϕ_0 say; if there are shock waves in the flow, then a coordinate straining also must be used. The analysis presented in this section for two-dimensional flow assumes, for simplicity, that there is not more than one shock wave on either surface of the airfoil and that any shock wave can be considered normal to the freestream. Although these restrictions are not fundamental to the basic concept, they do simplify the analysis.

The potential $\phi(x,z)$ is expanded as a series in some small parameter ϵ such as

$$\phi(x,z)=\phi_0(x',z)+\epsilon\phi_1(x',z)+\ldots \tag{8}$$

where x' is the strained coordinate. (Since the shock is assumed normal to the freestream, only the x coordinate need be strained.) The basic principles guiding the choice of x' are as follows.

If there is a shock wave on the upper (lower) surface of the airfoil, then the coordinate system in the upper (lower) half-plane is strained such that the shock remains in the same position in the new system throughout the perturbation. If there is no shock present in one half-plane, then no coordinate straining is necessary. Following the suggestion of Ref. 1, the strained coordinate system is defined by

$$x = x' + \epsilon \delta x_s x_1(x') + \ldots \tag{9}$$

where, if x_s' is the location of the shock in the (x', z) coordinates, then

$$x_1(x') = \frac{x'(1-x')}{x_s'(1-x_s')}; \quad 0 \le x' \le 1 \tag{10a}$$

$$x_1(x') = 0; \quad x' > 1, \quad x' < 0 \tag{10b}$$

$\epsilon \delta x_s$ is the amount by which the shock moves during the perturbation. If no shock is present, then

$$x_1(x') = 0$$

It is pointed out in Ref. 1 that the particular form of the straining function $x_1(x')$ is fairly arbitrary; the effect of differing straining functions is discussed in the Appendix. Substitution of Eqs. (8) and (9) into Eqs. (1) and (6) and equating coefficients of ϵ gives the equations

$$\phi_{0_{x'x'}} + \phi_{0_{zz}} = \phi_{0_{x'}} \phi_{0_{x'x'}} \tag{11}$$

with the boundary condition

$$\phi_{0_z}(x', +0) = Z_{u_{x'}}(x') \tag{12a}$$

$$\phi_{0_z}(x', -0) = Z_{L_{x'}}(x') \tag{12b}$$

and

$$\phi_{1_{x'x'}} + \phi_{1_{zz}} = (\phi_{1_{x'}} \phi_{0_{x'}})_{x'} + \delta x_s \{ [(\phi_{0_{x'}} - \phi_{0_{x'}}^2) x_{1_{x'}}]_{x'} + x_{1_{x'}} [\phi_{0_{x'}} - (\phi_{0_{x'}}^2/2)]_{x'} \} \tag{13}$$

with the boundary conditions

$$\phi_{1_z}(x', +0) = Z_{up_{x'}}(x') + \delta x_s x_1(x') Z_{u_{x'x'}}(x') \tag{14a}$$

$$\phi_{1_z}(x', -0) = Z_{Lp_{x'}}(x') + \delta x_s x_1(x') Z_{L_{x'x'}}(x') \tag{14b}$$

Since the magnitude of the perturbation ϵ does not appear in the linear equations, Eqs. (13) and (14), it follows that, once any values of ϕ_1, δx_s, and related variables are known for some magnitude of the perturbation ϵ_0, then the value of the variable for any other ϵ can be found by simple proportion. Hence, for example,

$$\epsilon \phi_{1_{x'}}(x', z) = [(\epsilon/\epsilon_0) \epsilon_0 \phi_{1_{x'}}(x', z)] \tag{15a}$$

$$\epsilon \delta x_s = (\epsilon/\epsilon_0)(\epsilon_0 \delta x_s) \tag{15b}$$

Although the linear perturbation equation, Eq. (13), can be solved for $\phi_{1_{x'}}$, δx_s, it is much more convenient to use the same technique to solve both the base and calibration solutions. Equation (13) multiplied by ϵ_0 represents, to first order of magnitude in ϵ_0, the equation satisfied by the difference of two solutions to Eq. (1). Hence, expressions for $\phi_{1_{x'}}$ and δx_s in Eq. (13) can be found by a suitable combination of two known nonlinear results.

If, by some method, the solution to the base problem defined by Eqs. (11) and (12) is known and if the solution to some perturbed problem, characterized by some parameter ϵ_0, is also known (the calibration solution), then the terms $(\epsilon_0 \phi_{1_{x'}})$, $(\epsilon_0 \delta x_s)$ can be found as follows:

1) The change in the shock location $\epsilon_0 \delta x_s$ between the base and calibration solutions is found easily by inspection.

2) If $\phi_{\hat{x}}^{(1)}(x,z)$ is the solution of the perturbed problem and if $\phi_{x'}^{(0)}(x',z)$ is the solution of the base problem, then

$$\epsilon_0 \phi_{1_{x'}}(x', z) = \phi_{\hat{x}}^{(1)}(\hat{x}, z) - \phi_{x'}^{(0)}(x', z)[1 - \epsilon_0 \delta x_s x_{1_{x'}}(x')] \tag{16}$$

where

$$\hat{x} = x' + \epsilon_0 \delta x_s x_1(x') \tag{17}$$

and $x_1(x')$ is given by Eq. (10).

Having obtained δx_s and $\phi_{1_x}(x', z)$, one then can obtain the final solution

$$\phi_x(x,z) = \phi_{x'}^{(0)}(x',z)[1 - \epsilon \delta x_s x_{1_{x'}}(x')] + \epsilon \phi_{1_{x'}}(x', z) \tag{18}$$

and

$$x = x' + \epsilon \delta x_s x_1(x') \tag{19}$$

If there are no shock waves in the flow, then

$$\delta x_s = 0$$

and Eqs. (18) and (19) reduce to the usual form of a perturbation solution in physical coordinates. Having obtained $\phi_{\hat{x}}(x,z)$, one can obtain the pressure coefficient $C_p(x,z)$ from Eq. (5).

B. Applications in Two-Dimensional Flow

The method just outlined can be used to compute the pressures around airfoils for different values of a perturbation parameter ϵ, provided that the base and calibration solutions are known. Some examples are given below.

The variation of the pressure distribution with freestream Mach number for a biconvex airfoil is given in Ref. 1. For completeness, it is presented again. Also, the change in pressure distribution with variations in angle of attack for a lifting airfoil with a shock-free lower surface is given.

The analytic nature of the relations in Eqs. (18) and (19) leads to simple formulas for the variation of the lift, drag, and pitching moment coefficients with a perturbation parameter. The derivation of these relations, together with examples, is presented.

In all of the computations, the base and calibration results are computed by using a conservative shock-capturing finite-difference program. Although the theory requires a discontinuous representation of a shock, it will be seen that the theory works satisfactorily for the commonly used shock-capture methods. The exponent q in the transonic parameter k in Eq. (4) is taken to be 2.0.

Variation of Pressure with Mach Number

Referring to Eq. (7), one obtains the relations

$$Z_{up}(x') = Z_u(x'), \quad Z_{Lp}(x') = Z_L(x') \tag{20}$$

and, using Eq. (4), one has

$$\epsilon = \left[\frac{(\gamma+1)M_\infty^q}{(1-M_\infty^2)^{3/2}} - \frac{(\gamma+1)M_0^q}{(1-M_0^2)^{3/2}} \right] \Big/ \left[\frac{(\gamma+1)M_0^q}{(1-M_0^2)^{3/2}} \right] \tag{21}$$

where M_0 is the freestream Mach number of the base flow; also,

$$\epsilon_0 = \left[\frac{(\gamma+1)M_1^q}{(1-M_1^2)^{3/2}} - \frac{(\gamma+1)M_0^q}{(1-M_0^2)^{3/2}} \right] \Big/ \left[\frac{(\gamma+1)M_0^q}{(1-M_0^2)^{3/2}} \right] \tag{22}$$

where M_1 is the freestream Mach number of the calibration flow. It can be verified by substitution of Eqs. (20) and (21) into Eq. (6) that this gives the correct variation with freestream Mach number.

The range of pressure distributions for several Mach numbers is shown in Fig. 1. The base Mach number is 0.828, and the calibration Mach number is 0.838. It can be seen that the agreement of the present theory with the direct finite-difference solution is very satisfactory.

Variation of Pressure with Angle of Attack

In this case, for an angle of attack α one has the relations

$$Z_{up_{x'}}(x') = -(k/\beta^3) \tag{23a}$$

$$Z_{Lp_{x'}}(x') = -(k/\beta^3) \tag{23b}$$

and

$$\epsilon = \alpha - \alpha_0 \tag{24a}$$

$$\epsilon_0 = \alpha_1 - \alpha_0 \tag{24b}$$

where α_0 and α_1 are the angles of attack of the base and calibration solutions, respectively. Again, it can be verified by substitution of Eqs. (23) and (24a) into Eq. (6) that this represents the correct variation with angle of attack.

The change in pressure was computed for the flow around an NACA 64A410 airfoil at $M_\infty = 0.74$ with $\alpha_0 = 1$ deg and $\alpha_1 = \frac{1}{2}$ deg. In this configuration, the lower surface of the airfoil is shock-free, and, hence, the coordinate straining is applied only in the upper half-plane. The results for angles of attack of 0 and 1.5 deg are shown in Fig. 2. The base solution also is shown for comparison. Again, as in the previous example, the results of the present method agree satisfactorily with the finite-difference results.

Computation of Lift, Pitching Moment, and Drag Coefficients

The algebraic dependence of the pressure on some parameter ϵ is given by Eqs. (5, 18, and 19). Using the present theory, these relations can be used to derive expressions for the lift, pitching moment, and drag coefficients, three of the most important parameters used in design calculations.

Calculations of Lift Coefficient: The lift coefficient C_L is defined by

$$C_L = \int_0^1 \Delta C_p(\bar{x})\,d\bar{x} \tag{25}$$

where the operator Δ is defined for a function $f(x,z)$ by

$$\Delta f(\bar{x}) = f(\bar{x}, -0) - f(\bar{x}, +0) \tag{26}$$

Using the pressure relations, Eq. (5), one obtains

$$C_L = -2\frac{\beta^2}{k}\int_0^1 \Delta\phi_x(x)\,dx \tag{27}$$

Because of probable differing straining functions on the upper and lower surfaces of the airfoil, it is helpful to consider initially the contribution of the pressure on one surface to the lift. Hence, consider the integral

$$I = 2\frac{\beta^2}{k}\int_0^1 \phi_x(x,+0)\,dx \tag{28}$$

The velocity $\phi_x(x,+0)$ is given by Eqs. (16) and (18), and, after substitution in Eq. (28), the integral becomes

$$I = \frac{2\beta^2}{k}\left(\int_0^1 \phi_{x'}{}^{(0)}(x',+0)[1-\epsilon\delta x_s x_{1_{x'}}(x')]dx + \frac{\epsilon}{\epsilon_0}\int_0^1 \left\{\phi_{\hat{x}}{}^{(1)}(\hat{x},+0) - \phi_{x'}{}^{(0)}(x',+0) \times [1-\epsilon_0\delta x_s x_{1_{x'}}(x')]\right\}dx\right) \tag{29}$$

where

$$\hat{x} = x' + \epsilon_0 \delta x_s x_1(x') \tag{30}$$

From Eq. (19), it can be seen that

$$dx = [1+\epsilon\delta x_s x_{1_{x'}}(x')]dx' \tag{31a}$$

and, from Eqs. (30) and (31a),

$$dx = [1+(\epsilon-\epsilon_0)\delta x_s x_{1_{\hat{x}}}(\hat{x})]d\hat{x} \tag{31b}$$

Substitution of Eq. (31) into Eq. (29) gives

$$I = \frac{2\beta^2}{k}\left\{\int_0^1 \phi_{x'}{}^{(0)}(x',+0)dx' + \frac{\epsilon}{\epsilon_0}\left[\int_0^1 \phi_{\hat{x}}{}^{(1)}(\hat{x},+0)d\hat{x} - \int_0^1 \phi_{x'}{}^{(0)}(x',+0)dx'\right] + \frac{\epsilon}{\epsilon_0}(\epsilon-\epsilon_0)\delta x_s(A_1 - A_0)\right\} \tag{32}$$

where

$$A_0 = \int_0^1 \phi_{x'}{}^{(0)}(x',+0)x_{1_{x'}}(x')$$

$$A_1 = \int_0^1 \phi_{\hat{x}}{}^{(1)}(\hat{x},+0)x_{1_{\hat{x}}}(\hat{x})d\hat{x}$$

Now by definition

$$\left[\int_0^1 \phi_{\hat{x}}{}^{(1)}(\hat{x},+0)d\hat{x} - \int_0^1 \phi_{x'}{}^{(0)}(x',+0)dx'\right] \sim O(\epsilon_0)$$

$$(A_1 - A_0) \sim O(\epsilon_0)$$

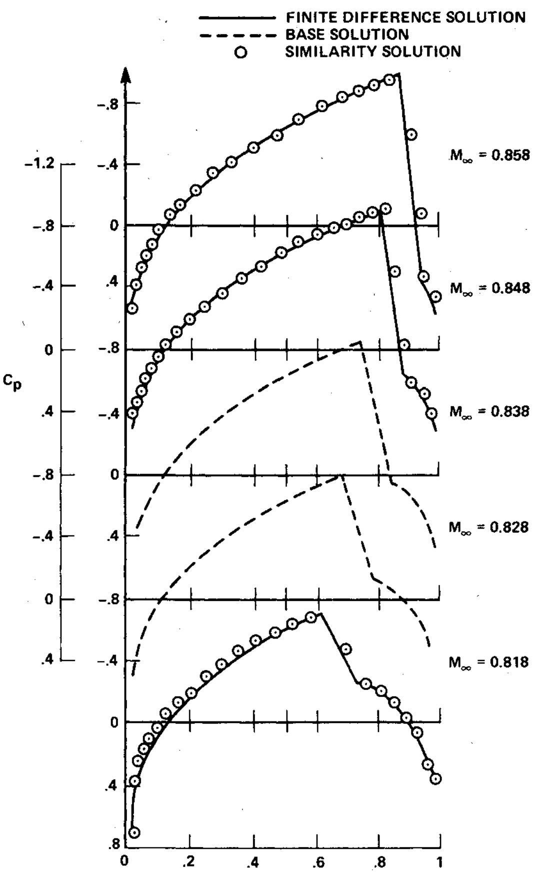
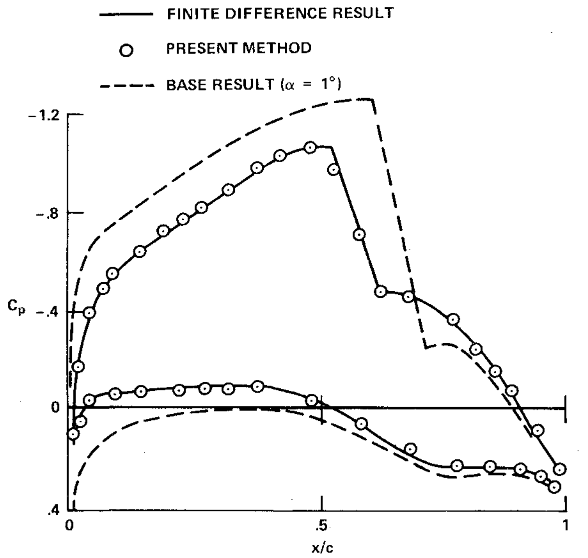
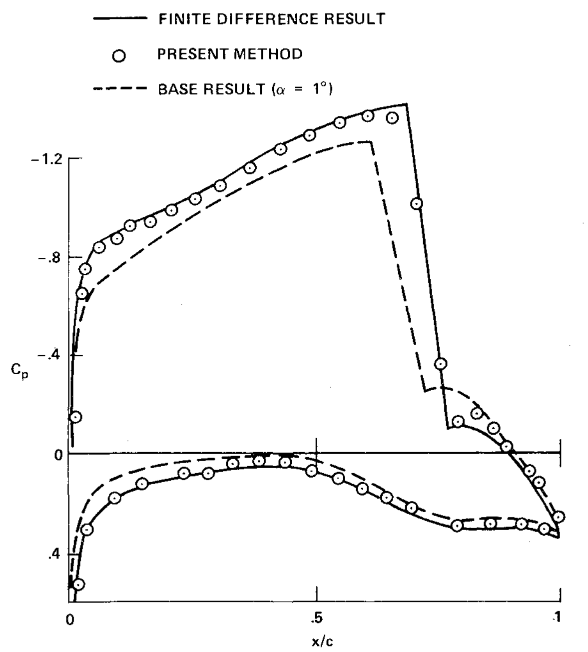


Fig. 1 Variation of pressure with Mach number for a 10% parabolic-arc airfoil.



a) $\alpha = 0$ deg



b) $\alpha = 1.5$ deg

Fig. 2 Pressure distribution around an NACA 64A410 airfoil ($M_\infty = 0.74$).

and, hence, to first order in ϵ, Eq. (32) reduces to

$$I = \frac{2\beta^2}{k}\left\{\int_0^1 \phi_{x'}{}^{(0)}(x',+0)\mathrm{d}x' + \frac{\epsilon}{\epsilon_0}\left[\int_0^1 \phi_{\hat{x}}{}^{(1)}(\hat{x},+0)\mathrm{d}\hat{x} - \int_0^1 \phi_{x'}{}^{(0)}(x',+0)\mathrm{d}x'\right]\right\} \tag{33}$$

Application of a similar procedure to the lower surface pressures and addition of the two results gives

$$C_L = \frac{\beta^2}{k}\left\{\left(\frac{k}{\beta^2}\right)^{(0)} C_{L_0} + \frac{\epsilon}{\epsilon_0}\left[\left(\frac{k}{\beta^2}\right)^{(1)} C_{L_1} - \left(\frac{k}{\beta^2}\right)^{(0)} C_{L_0}\right]\right\} \tag{34}$$

where C_{L_0}, C_{L_1} are the lift coefficients of the base and calibration solutions, respectively. The terms $(k/\beta^2)^{(0)}$ and $(k/\beta^2)^{(1)}$ refer to conditions for the base and calibration solutions, respectively.

If the freestream Mach number is unchanged by the perturbation, then

$$C_L = C_{L_0} + (\epsilon/\epsilon_0)(C_{L_1} - C_{L_0}) \tag{35}$$

Calibration of Pitching Moment Coefficient: The pitching moment about the leading edge, C_m, is defined by

$$C_m = -\int_0^1 \Delta C_p(\bar{x})\bar{x}\mathrm{d}\bar{x} \tag{36}$$

where the operator Δ is defined by Eq. (26), and nose-up pitching moments are positive. Only pitching moments about the leading edge are considered because of simplicity in presentation; the moment about any other axis is obtained easily by a suitable combination of C_L and C_m.

If a similar procedure to that given for the lift coefficient is applied to Eq. (36), then the resulting relation for C_m is

$$C_m = \frac{\beta^2}{k}\left\{\left(\frac{k}{\beta^2}\right)^{(0)} C_{m_0} + \frac{\epsilon}{\epsilon_0}\left[\left(\frac{k}{\beta^2}\right)^{(1)} C_{m_1} - \left(\frac{k}{\beta^2}\right)^{(0)} C_{m_0}\right]\right\} \tag{37}$$

where C_{m_0}, C_{m_1} are the pitching moment coefficients of the base and calibration solutions, respectively. If the freestream Mach number is not perturbed, then

$$C_m = C_{m_0} + (\epsilon/\epsilon_0)(C_{m_1} - C_{m_0}) \tag{38}$$

Calculation of Drag Coefficient: Two methods of calculating the drag coefficient are considered: 1) the integration of the drag component of the surface pressures, and 2) the relation derived by Murman and Cole.[2] If the drag coefficient C_D is to be estimated by integrating the drag component of the pressure over the airfoil, then

$$C_D = -\int_0^1 [C_p(\bar{x},+0)F'_+(\bar{x}) - C_p(\bar{x},-0)F'_-(\bar{x})]\mathrm{d}\bar{x} \tag{39}$$

where $F'_+(x)$ and $F'_-(x)$ are the slopes of the upper and lower surfaces of the airfoil, respectively. If an analysis similar to that used for the lift coefficient is used, then Eq. (39) becomes

$$C_D = \frac{\beta^2}{k}\left\{\left(\frac{k}{\beta^2}\right)^{(0)} C_{D_0} + \frac{\epsilon}{\epsilon_0}\left[\left(\frac{k}{\beta^2}\right)^{(1)} C_{D_1} - \left(\frac{k}{\beta^2}\right)^{(0)} C_{D_0}\right]\right\} \tag{40}$$

where C_{D_0} and C_{D_1} are the drag coefficients of the base and calibration solutions, respectively.

The drag relationship derived by Murman and Cole[2] can be written in the present variables as

$$C_D = -\frac{\gamma+1}{6M_\infty^2}\,\frac{1}{\beta}\left(\frac{\beta^2}{k}\right)^3 \int_{\text{shocks}} \{[\phi_x(x,z)]_-^+\}^3 \mathrm{d}z \tag{41}$$

where $[\ ]_-^+$ denotes a jump across the shock wave. The integral is performed along all shock waves in the flow.

The basic perturbation theory requires that the shock should always be at the same location. Since the lengths of the shocks in the base and calibration solutions are unlikely to be equal, the z coordinate also should be strained so that the shock end points coincide; the transformation used is

$$z = z' + \frac{\epsilon\delta z_T}{z'_T} z' \tag{42}$$

where z_T' is the location of the shock end point of the base solution in the (x',z') coordinates, and $\epsilon\delta z_T$ is a measure of the change in shock length.

The velocity at the shock can be obtained to first order in ϵ from Eqs. (16) and (18); thus,

$$\phi_x(x_s,z) = \phi_{x'}^{(0)}(x_s',z') + (\epsilon/\epsilon_0)[\phi_{\hat{x}}^{(1)}(\hat{x}_s,\hat{z}) - \phi_{x'}^{(0)}(x_s',z')] \tag{43}$$

where z is given by Eq. (42) and

$$\hat{x} = x' + \epsilon_0 \delta x_s x_1(x') \tag{44a}$$

$$\hat{z} = z' + \epsilon_0 \frac{\delta z_T}{z_T'} z' \tag{44b}$$

Substitution of Eqs. (42-44) into Eq. (41) gives

$$C_D = -\frac{\gamma+1}{6M_\infty^2}\frac{1}{\beta}\left(\frac{\beta^2}{k}\right)^3\left[a_0 + a_1\frac{\epsilon}{\epsilon_0} + a_2\left(\frac{\epsilon}{\epsilon_0}\right)^2 + a_3\left(\frac{\epsilon}{\epsilon_0}\right)^3\right] \tag{45}$$

where

$$a_0 = \int_{\text{shocks}} \left\{\left[\phi_{x'}^{(0)}(x_s',z')\right]_-^+\right\}^3 dz' \tag{46a}$$

$$a_1 = 3\int_{\text{shocks}} \left\{\left[\phi_{x'}^{(0)}(x_s',z')\right]_-^+\right\}^2 \left\{\left[\phi_{\hat{x}}^{(1)}(\hat{x}_s,\hat{z}) - \phi_{x'}^{(0)} \times (x_s',z')\right]_-^+\right\} dz' + \int_{\text{shocks}} \left\{\left[\phi_{x'}^{(0)}(x_s',z')\right]_-^+\right\}^3 \frac{\delta z_T}{z_T'} dz' \tag{46b}$$

$$a_2 = 3\int_{\text{shocks}} \left\{\left[\phi_{x'}^{(0)}(x_s',z')\right]_-^+\right\}\left\{\left[\phi_{\hat{x}}^{(1)}(\hat{x}_s,\hat{z}) - \phi_{x'}^{(0)}(x_s',z')\right]_-^+\right\}^2 dz' \tag{46c}$$

$$a_3 = \int_{\text{shocks}} \left\{\left[\phi_{\hat{x}}^{(1)}(\hat{x}_s,\hat{z}) - \phi_{x'}^{(0)}(x_s',z')\right]_-^+\right\}^3 dz' \tag{46d}$$

Examples: Examples of the calculation of lift, pitching moment, and drag are shown in Figs. 3 and 4. In Figs. 3a and 3b, the lift and pitching moment variation with angle of attack for an NACA 64A410 at $M_\infty = 0.74$ are compared with direct finite-difference calculations. The base and calibration solutions are at $\alpha = 0$ deg and $\alpha = ½$ deg, respectively. The agreement between the results is adequate.

In Fig. 4, the variation of the drag coefficient with freestream Mach number for a 10% biconvex airfoil at zero angle of attack, using both Eqs. (40) and (45), is compared with the finite-difference result. The base and calibration Mach numbers are 0.808 and 0.818, respectively. The finite-difference drag is computed by direct integration of the pressures around the airfoil. It can be seen that Eq. (40) gives results that are in poor agreement with the finite-difference results, even for small changes in M_∞ whereas Eq. (45) gives much more satisfactory agreement. This implies that the region of validity of Eq. (40) is much less than that of Eq. (45).

In the foregoing examples, the base and calibration solutions have been computed for parameters at one extreme of the range presented. This was done in order to indicate the kind of errors which may be expected from an arbitrary choice of base and calibration solutions.

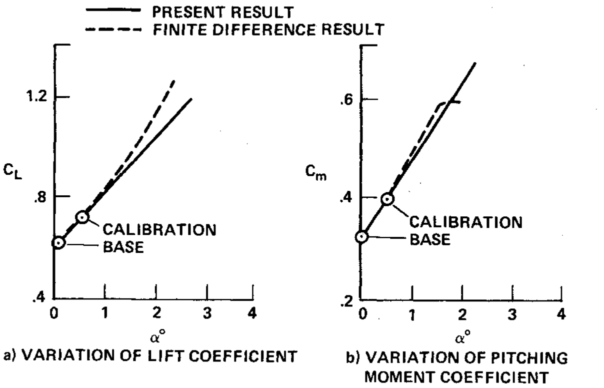


Fig. 3 Variation of lift and pitching moment coefficients with angle of attack (NACA 64A410 airfoil, $M_\infty = 0.74$).

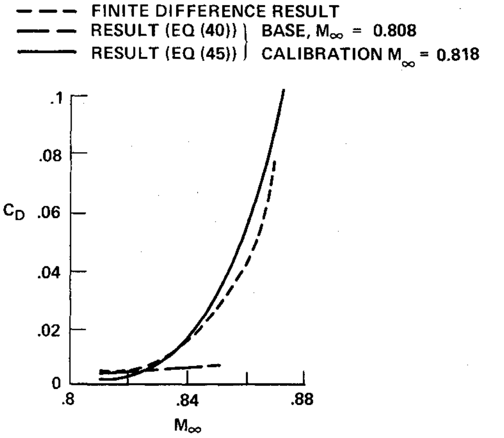


Fig. 4 Variation of drag coefficient with Mach number for a 10% parabolic-arc airfoil.

IV. Three-Dimensional Flows

The main differences between two- and three-dimensional flows are in the basic small disturbance equation used and in the type of flows encountered. An example of the latter is the more common occurrence in three-dimensional flow of multiple, intersecting shock waves; also shock waves may not stretch the full span, and the shock end points may change during the perturbation. The theory outlined earlier for two-dimensional flows therefore must be developed to treat a different potential equation and to treat multiple, intersecting shock waves.

A. Basic Equations

The basic potential equation used for the three-dimensional analysis is the small disturbance equation used by Ballhaus et al.,[3] namely,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = \phi_x\phi_{xx} + \frac{(3-\gamma)}{2k}\beta^2 M_\infty^2(\phi_y^2)_x + \frac{(\gamma-1)M_\infty^2\beta^2}{k}(\phi_x\phi_y)_y \tag{47}$$

where (x,y,z) is a Cartesian coordinate system with the origin at the apex of the wing and is related to the physical coordinate system $(\bar{x},\bar{y},\bar{z})$ by

$$x = \bar{x}, \quad y = \beta\bar{y}, \quad z = \beta\bar{z} \tag{48}$$

ϕ is related to the physical perturbation potential $\bar{\phi}$ by

$$\phi(x,y,z) = (k/\beta^2)\bar{\phi}(\bar{x},\bar{y},\bar{z}) \tag{49}$$

and k is given by Eq. (4).

The thin-wing boundary conditions appropriate for a perturbation of Eq. (47) are

$$\phi_z(x,y,+0) = Z_{u_x}(x,y) + \epsilon Z_{up_x}(x,y) \quad (50a)$$

$$\phi_z(x,y,-0) = Z_{L_x}(x,y) + \epsilon Z_{Lp_x}(x,y) \quad (50b)$$

where, if $\bar{z}=\bar{z}_u(\bar{x},\bar{y})$ and $\bar{z}=\bar{z}_L(\bar{x},\bar{y})$ are the geometries of the upper and lower surfaces of the wing, respectively, then

$$Z_u(x,y) = (k/\beta^3)\bar{z}_u(\bar{x},\bar{y}) \quad (51a)$$

$$Z_L(x,y) = (k/\beta^3)\bar{z}_L(\bar{x},\bar{y}) \quad (51b)$$

$Z_{up}(x,y)$ and $Z_{Lp}(x,y)$ denote the form of a perturbation to the wing surface.

As mentioned in previous sections, the basic idea is to linearize the potential equation, Eq. (47), about some known solution ϕ_0. If there are shock waves in the flow, then a strained coordinate system is used in which the shock locations are invariant. It is assumed here that any shock waves are normal to the wing planform, and, hence, only the coordinates x,y need be strained.

The potential $\phi(x,y,z)$ is expanded as

$$\phi(x,y,z) = \phi_0(x',y',z) + \epsilon\phi_1(x',y',z) + \ldots \quad (52)$$

where x', y' are distorted coordinates. Allowing for N shock waves in the flow, the distorted coordinates x',y' are given by

$$x = x' + \sum_{i=1}^{N} \epsilon\delta x_{s_i}(y')x_{1_i}(x',y') \quad (53a)$$

$$y = y' + \sum_{i=1}^{N} \epsilon\delta y_{T_i} y_{1_i}(y') \quad (53b)$$

where $\epsilon\delta x_{s_i}(y')$ is the change in location of the ith shock at a spanwise station y'; δy_{T_i} is the change in location of the end point of the ith shock. Unlike the two-dimensional case discussed in Sec. III, the basic equation, Eq. (47) is not invariant, since it can be seen that, if the freestream Mach number is perturbed, then the coefficients of certain terms can change. For perturbations of M_∞, then, the perturbation equation may be nonlinearly dependent on ϵ, thus invalidating the similarity theory. Hence, some means of approximating changes in the Mach number dependent terms by a linear function of ϵ must be found.

If there is a perturbation in Mach number, then, as in Eq. (21),

$$\epsilon = \frac{(k/\beta^3)}{(k/\beta^3)^{(0)}} - 1 \quad (54)$$

The Mach number dependence of Eq. (47) appears through the term $M_\infty^2\beta^2/k$, which can be written as

$$\frac{M_\infty^2\beta^2}{k} = \frac{M_\infty}{k^{1/3}}\left(\frac{\beta^3}{k}\right)^{2/3}$$

which, using Eqs. (4), becomes

$$\frac{M_\infty^2\beta^2}{k} = \frac{M_\infty^2}{[(\gamma+1)M_\infty^q]^{1/3}}\left(\frac{\beta^3}{k}\right)^{2/3} \quad (55)$$

Substitution of Eq. (54) into Eq. (55) gives

$$\frac{M_\infty^2\beta^2}{k} = \frac{M_\infty^2}{[(\gamma+1)M_\infty^q]^{1/3}}\left(\frac{\beta^2}{k^{2/3}}\right)^{(0)}(1-2/3\epsilon+\ldots) \quad (56)$$

A basic premise of the transonic small disturbance theory leading to Eq. (47) is that changes in M_∞ are of an order of magnitude less than changes in β^2. Hence, the change in $M_\infty^2\beta^2/k$ due to a change in M_∞ can be approximated by

$$\frac{M_\infty^2\beta^2}{k} = \frac{M_0^2}{[(\gamma+1)M_0^q]^{1/3}}\left(\frac{\beta^2}{k^{2/3}}\right)^{(0)}(1-2/3\epsilon+\ldots)$$

$$= M_0^2\left(\frac{\beta^2}{k}\right)^{(0)}(1-2/3\epsilon+\ldots) \quad (57)$$

Substitution of Eqs. (52, 53, and 57) into Eqs. (47) and (50) and equating coefficients of ϵ gives

$$\phi_{0_{x'x'}} + \phi_{0_{y'y'}} + \phi_{0_{zz}} = \phi_{0_{x'}}\phi_{0_{x'x'}} + \frac{3-\gamma}{2}\left(\frac{M_\infty^2\beta^2}{k}\right)^{(0)}(\phi_{0_{y'}}^2)_{x'} + (\gamma-1)\left(\frac{M_\infty^2\beta^2}{k}\right)^{(0)}(\phi_{0_{x'}}\phi_{0_{y'}})_{y'} \quad (58)$$

with the boundary conditions

$$\phi_{0_z}(x_i'y_i'+0) = Z_{u_x}(x',y') \quad (59a)$$

$$\phi_{0_z}(x_i'y_i'-0) = Z_{L_{x'}}(v',y) \quad (59b)$$

and

$$\phi_{1_{x'x'}} + \phi_{1_{y'y'}} + \phi_{1_{zz}} = (\phi_{0_{x'}}\phi_{1_{x'}})_{x'} + \left(\frac{M_\infty^2\beta^2}{k}\right)^{(0)}\Big[(3-\gamma)(\phi_{1_{y'}}\phi_{0_{y'}})_{x'} + (\gamma-1)(\phi_{0_{x'}}\phi_{1_{y'}} + \phi_{1_{x'}}\phi_{0_{y'}})_{y'}\Big]$$
$$+ \sum_{i=1}^{N}\left\{\delta x_{s_i}(y')\Phi_i^{(1)} + \frac{\partial}{\partial y'}\left[\delta x_{s_i}(y')\right]\Phi_i^{(2)}\right\} + \sum_{i=1}^{N}\delta y_{T_i}\Phi_i^{(3)} + \Phi^{(4)} \quad (60)$$

where

$$\Phi_i^{(1)} = [x_{1_{i_{x'}}}(x',y')(\phi_{0_{x'}} - \phi_{0_{x'}}^2)]_{x'} + x_{1_{i_{x'}}}(x',y')\left(\phi_{0_{x'}} - \frac{\phi_{0_{x'}}^2}{2}\right)_{x'} - \left(\frac{M_\infty^2\beta^2}{k}\right)^{(0)}$$
$$\times\left\{\frac{3-\gamma}{2}x_{1_{i_{x'}}}(x',y')(\phi_{0_{y'}}^2)_{x'} + (\gamma-1)\Big[(\phi_{0_{y'}}\phi_{0_{x'}})_{y'}x_{1_{i_{x'}}}(x',y') + \phi_{0_{x'}}\phi_{0_{y'}}x_{1_{i_{x'y'}}}(x',y')\Big]\right\} \quad (61a)$$

$$\Phi_i^{(2)} = -(\gamma-1)\left(\frac{M_\infty^2\beta^2}{k}\right)^{(0)}x_{1_{i_{x'}}}(x',y')(\phi_{0_{y'}}\phi_{0_{x'}}) \quad (61b)$$

$$\Phi_i^{(3)} = 2y_{1_{i_{y'}}}(y')\phi_{0_{y'y'}} + y_{1_{i_{y'y'}}}\phi_{0_{y'}} - \left(\frac{M_\infty^2\beta^2}{k}\right)^{(0)}\Big[(3-\gamma)y_{1_{i_{y'}}}(y')(\phi_{0_{y'}}^2)_{x'}$$

$$+(\gamma-1)(\phi_{0_{x'}}\phi_{0_{y'}})y_{1_{i_{y'y'}}}(y') + 2(\gamma-1)(\phi_{0_{y'}}\phi_{0_{x'}})_{y'}y_{1_{i_{y'}}}(y')\Big] \tag{61c}$$

$\Phi_i^{(4)} = 0$ if M_∞ const

$$= -\frac{2}{3}\left(\frac{M_\infty^2\beta^2}{k}\right)^{(0)}\left[\frac{3-\gamma}{2}(\phi_{0_{y'}}^2)_{x'} + (\gamma-1)(\phi_{0_{x'}}\phi_{0_{y'}})_{y'}\right] \text{ if } M_\infty \text{ perturbed} \tag{61d}$$

The boundary conditions for Eq. (60) are

$$\phi_{1_z}(x',y',+0) = Z_{up_{x'}}(x') + \left[\sum_{i=1}^{N}\delta x_{s_i}(y')x_{1_i}(x',y')\right]Z_{u_{x'x'}}(x',y') \tag{62a}$$

$$\phi_{1_z}(x',y',-0) = Z_{Lp_{x'}}(x') + \left[\sum_{i=1}^{N}\delta x_{s_i}(y')x_{1_i}(x',y')\right]Z_{L_{x'x'}}(x',y') \tag{62b}$$

As in the case of two-dimensional flow, it can be seen that the magnitude of the perturbation ϵ does not appear in Eqs. (60) and (62). Hence, if the values of $\delta x_{s_i}(y')$, δy_{T_i}, ϕ, or related variables are known for some value ϵ_0, then the values of these variables for any other ϵ can be obtained from the simple relationship

$$\frac{(\epsilon\delta x_{s_i})^{(1)}}{(\epsilon\delta x_{s_i})^{(0)}} = \frac{\epsilon}{\epsilon_0} = \frac{(\epsilon\delta y_{T_i})^{(1)}}{(\epsilon\delta y_{T_i})^{(0)}} = \frac{(\epsilon\phi_{1_{x'}})^{(1)}}{(\epsilon\phi_{1_{x'}})^{(0)}} \tag{63†}$$

with

$$\phi_x(x,y,z) = \phi_{0_{x'}}(x',y',z)\left[1-\epsilon\sum_{i=1}^{N}\delta x_{s_i}(y')x_{1_{i_{x'}}}(x',y')\right] + \epsilon\phi_{1_{x'}}(x',y',z) \tag{64}$$

and

$$x = x' + \epsilon\sum_{i=1}^{N}\delta x_{s_i}(y')x_1(x',y') \tag{65a}$$

$$y = y' + \epsilon\sum_{i=1}^{N}\delta y_{T_i}y_1(x',y') \tag{65b}$$

Having found $\phi_x(x,y,z)$, one can find the pressure coefficient by using Eq. (5). The method for determining δy_{T_i}, δx_{s_i}, $(\phi_{1_{x'}})$ is the same as that outlined in Sec. III.A. Thus, the linear Eq. (60) multiplied by ϵ_0 is regarded as an approximation to the difference between two known nonlinear results, which are computed using standard methods.[3] Thus we have the following:

1)The shock movements $\epsilon_0\delta y_{T_i}$, $\epsilon_0\delta x_{s_i}$ are obtained by inspection.

2) If $\phi_x^{(0)}(x',y',z)$ is the base solution and $\phi_x^{(1)}(\hat{x},\hat{y},z)$ is the calibration solution, then

$$\epsilon_0\phi_{1_x}(x',y',z) = \phi_{\hat{x}}^{(1)}(\hat{x},\hat{y},z) - \phi_{x'}^{(0)}(x',y',z)\left[1-\epsilon\sum_{i=1}^{N}\delta x_{s_i}(y')x_{1_{i_{x'}}}(x',y')\right]$$

where

$$\hat{x} = x' + \epsilon_0\sum_{i=1}^{N}\delta x_{s_i}(y')x_{1_i}(x',y')$$

$$\hat{y} = y' + \epsilon_0\sum_{i=1}^{N}\delta y_{T_i}y_{1_i}(y')$$

B. Choice of Straining

Single Full-Span Shock

The simplest three-dimensional problem is that of a single shock wave on a surface of a wing with the shock extending the full length of the span. In this case, we have

$$x_1(x',y') = \frac{[x_L(y')-x'][x_T(y')-x']}{[x_L(y')-x_s'(y')][x_T(y')-x_s'(y')]} \text{ on wing} \tag{66a}$$

$$x_1(x',y') = 0 \quad \text{off wing} \tag{66b}$$

$$y_1(y') = 0 \tag{66c}$$

†All of the variables in Eq. (60) are related to ϵ through similar formulas.

where $x'_s(y')$ is the shock location at the spanwise stations y', and $x_L(y')$, $x_T(y')$ denote the leading and trailing edge of the wing, respectively, at the spanwise station y'.

Two Intersecting Shocks

In this case, one of the shocks terminates inboard of the wing tip at the point of intersection of the shocks. If the point of intersection is denoted by y'_T, then a suitable straining is

$$x_{1_1}(x',y') = \frac{[x_L(y')-x'][x_T(y')-x'][x'-x'_{s_2}(y')]}{[x_L(y')-x'_{s_1}(y')][x_T(y')-x'_{s_1}(y')][x_{s_1}(y')-x'_{s_2}(y')]} \tag{67a}$$

$$x_{1_2}(x',y') = -\frac{[x_L(y')-x'][x_T(y')-x'][x'-x'_{s_1}(y')]}{[x_L(y')-x'_{s_2}(y')][x_T(y')-x'_{s_1}(y')][x'_{s_1}(y')-x'_{s_2}(y')]} \quad \text{on wing} \tag{67b}$$

$$x_{1_1}(x',y') = x_{1_2}(x',y') = 0 \quad \text{off wing} \tag{67c}$$

$$y_1(y') = \frac{\delta y_T y'(s-y')}{y'_T(s-y'_T)} \quad \text{on wing,} \quad y_1(y') = 0 \ \text{off wing} \tag{67d}$$

where $x'_{s1}(y')$, $x'_{s2}(y')$ are the locations of the shock waves in the base solution, and s is the semispan of the wing.

C. Application in Three-Dimensional Flow

The theory just outlined has been applied to the flow about the ONERA M6 wing in a configuration, which gives two intersecting shock waves. The base and calibration solutions are calculated by using a conservative finite-difference method with the exponent q in Eq. (4) equal to 1.75. The base solution is at $M_\infty = 0.84$, $\alpha = 4$ deg and the calibration solution at $M_\infty = 0.84$, $\alpha = 3$ deg. These two results were used to compute the pressure distribution at $M_\infty = 0.84$ and $\alpha = 5$ deg at certain spanwise stations, which are shown in Fig. 5, and the change in shock geometry, which is shown in Fig. 6. The agreement between the present results and the finite-difference results is satisfactory.

V. Perturbations of More Than One Parameter

A. Basic Theory

The perturbation equation, Eq. (60), is linear in ϕ_1, and, consequently, complex solutions can be constructed from the superposition of less complex solutions. As an illustration of this idea, consider a two-parameter problem in which the two perturbation parameters are denoted by ϵ and Δ, and let

$$\phi(x,y,z) = \phi_0(x',y',z) + \epsilon\phi_1^\epsilon(x',y',z) + \Delta\phi_1^\Delta(x',y',z) + \ldots \tag{68}$$

with the straining written as

$$x = x' + \sum_{i=1}^{N} (\epsilon\delta x_{s_i}^\epsilon + \Delta\delta x_{s_i}^\Delta) x_{1_i}(x',y') \tag{69a}$$

$$y = y' + \sum_{i=1}^{N} (\epsilon\delta y_{T_i}^\epsilon + \Delta\delta y_{T_i}^\Delta) y_{1_i}(y') \tag{69b}$$

where the superscripts ϵ, Δ refer to functions dependent on the perturbation parameters ϵ and Δ, respectively. Since ϵ, Δ are two independent parameters, they can be treated separately. Thus, from the base solution, a perturbation in ϵ gives a calibration solution for the ϵ-dependent terms, and a perturbation in Δ gives a calibration solution for the Δ-dependent terms. Thus,

$$\frac{(\epsilon\delta x_{s_i}^\epsilon)^{(1)}}{(\epsilon\delta x_{s_i}^\epsilon)^{(0)}} = \frac{\epsilon}{\epsilon_0} = \frac{(\epsilon\phi_{1_{x'}}^\epsilon)^{(1)}}{(\epsilon\phi_{1_{x'}}^\epsilon)^{(0)}} \tag{70a}$$

$$\frac{(\Delta\delta x_{s_i}^\Delta)^{(1)}}{(\Delta\delta x_{s_i}^\Delta)^{(0)}} = \frac{\Delta}{\Delta_0} = \frac{(\Delta\phi_{1_{x'}}^\Delta)^{(1)}}{(\Delta\phi_{1_{x'}}^\Delta)^{(0)}} \tag{70b}$$

where ϵ_0, Δ_0 are the values of ϵ, Δ characterizing the calibration solutions for ϵ and Δ, respectively. The total value of $\phi_x(x,y,z)$ then is given by

$$\phi_x(x,y,z) = \phi_{0_{x'}}(x',y',z)\left[1 - \epsilon\sum_{i=1}^{N}\delta x_{s_i}^\epsilon x_{1_{i_{x'}}}(x',y') - \Delta\sum_{i=1}^{N}\delta x_{s_i}^\Delta x_{1_{i_{x'}}}(x',y')\right] + \epsilon\phi_{1_x}^\epsilon(x',y',z) + \Delta\phi_{1_x}^\Delta(x',y',z) \tag{71}$$

where

$$x = x' + \epsilon\sum_{i=1}^{N}\delta x_{s_i}^\epsilon(y')x_{1_i}(x',y') + \Delta\sum_{i=1}^{N}\delta x_{s_i}^\Delta(y')x_{1_i}(x',y') \tag{72a}$$

$$y = y' + \epsilon\sum_{i=1}^{N}\delta y_{T_i}^\epsilon y_{1_i}(y') + \Delta\sum_{i=1}^{N}\delta y_{T_i}^\Delta y_{1_i}(y') \tag{72b}$$

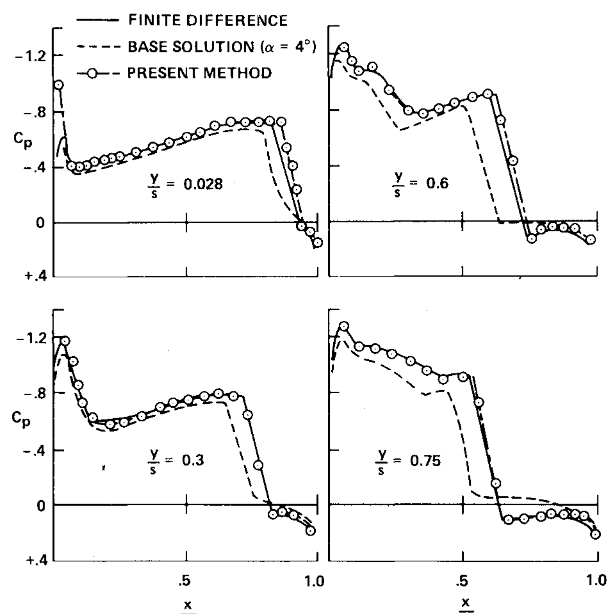


Fig. 5 Pressure distribution on the upper surface of ONERA M6 wing ($M_\infty = 0.84$, $\alpha = 5$ deg).

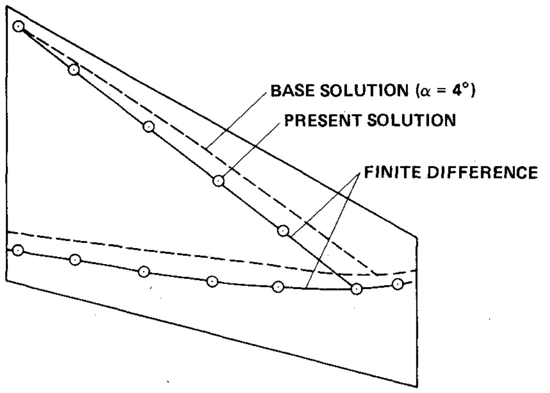


Fig. 6 Shock pattern on the ONERA M6 wing ($M_\infty = 0.84$, $\alpha = 5$ deg).

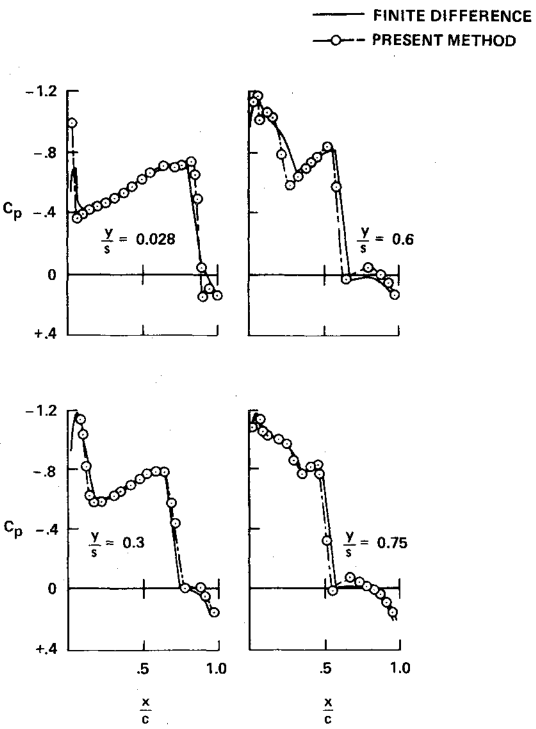


Fig. 7 Pressure distribution on the ONERA M6 wing ($M_\infty = 0.84$, $\alpha = 5$ deg, $\theta = -3$ deg).

Having found ϕ_x, one can obtain the pressure coefficient from Eq. (5). The method of obtaining the terms δx_s^ϵ, δx_s^A, ϕ_{1x}^ϵ, ϕ_{1x}^A is essentially the same as that presented in Sec. IV.A.

B. Applications

In order to test the applicability of a multiparameter perturbation, the flow around the ONERA M6 wing with perturbations in angle of attack and twist is examined. As in the example in Sec. IV, the base solution is at $M_\infty = 0.84$, $\alpha = 4$ deg, and $\theta = 0$ deg, where θ is the angle of twist. The calibration solution for angle of attack is at $M_\infty = 0.84$, $\alpha = 3$ deg, and $\theta = 0$ deg; for the twist it is at $M_\infty = 0.84$, $\alpha = 4$ deg, and $\theta = 1.5$ deg. These results were used to compute the flow at $M_\infty = 0.84$, $\alpha = 5$ deg, and $\theta = -3$ deg; the resulting pressure distribution is shown in Fig. 7. The agreement between the present result and the finite-difference result is fairly satisfactory. It is thought that the discrepancies are partly due to the large difference between the calibration twist angle ($\theta = 1.5$ deg) and the test twist angle ($\theta = -3$ deg), which may be near the limits of the validity of the perturbation.

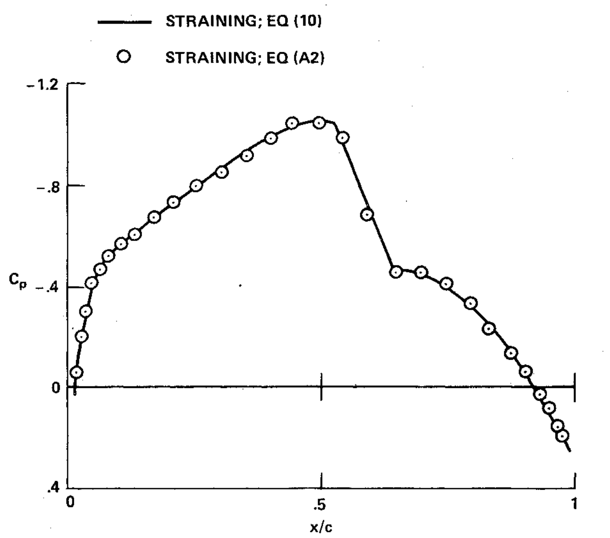


Fig. 8 Pressure distribution around the upper surface of an NACA 64A410 airfoil ($M_\infty = 0.74$, $\alpha = 0$ deg) for different straining functions.

VI. Concluding Remarks

A method of perturbing discontinuous transonic flows has been presented in which transonic flows are related through simple algebraic formulas. The theory is subject to the restriction that shock waves are not lost or generated during the perturbation process. It is shown how the method can be used to calculate the lift, drag, and pitching moment coefficients for two-dimensional flows, in addition to the pressure coefficient. It is felt that this gives a useful start to applications of the theory to design optimization problems in which the analytic nature of the present theory should be of considerable help.

Applications of the method to three-dimensional flows with intersecting shock waves and to flows where more than one parameter is perturbed also are shown, indicating the potentialities of the method. Since the computing times of the present method are of the order of 0.3 s in a CDC 7600 for the three-dimensional cases, it is suggested that the method is of considerable benefit in general transonic calculations, especially those for finite wings.

Appendix: Effect of Different Straining Functions

The straining function $[x_I(x')]$ used in the main part of the paper and suggested in Ref. 1 is chosen fairly arbitrarily; the only restrictions are that the straining vanish like x' as $x' \to 0$ and that the shock location is invariant. This arbitrariness raises the possibility of nonuniqueness of the solutions. Consequently, the effect of using different straining functions is investigated. The following analysis is presented for two-dimensional flows with a single shock wave, although, in principle, extensions to more complex cases are straightforward.

The alternative straining function is given by

$$x = x' + \epsilon \delta x_s x_I(x') \tag{A1}$$

where

$$x_I(x') = x'H(x_s' - x') + (I - x')H(x' - x_s'); \quad 0 \le x' \le I \tag{A2a}$$

$$x_I(x') = 0; \quad x' < 0, \ x' > I \tag{A2b}$$

and $H(y')$ is the step function, i.e.,

$$H(y') = 0; \quad y' < 0$$

$$H(y') = I; \quad y' > 0$$

In the velocity relation, Eq. (18), the derivative of $x_I(x')$ is required. Differentiation of Eq. (A2) gives

$$x_{I_{x'}}(x') = H(x_s' - x') - H(x' - x_s') + x'\delta(x_s' - x') + (I - x')\delta(x' - x_s') \quad \text{(A3)}$$

where $\delta(y)$ is the delta function. In Eq. (A3), the delta function is infinite at $x' = x_s'$, and the straining does not apply at the shock. However, since the velocity actually at the shock usually is not required, this not a great handicap for this form of straining.

The pressure distribution over the upper surface of an NACA 64A410 at $\alpha = 0$ deg and $M_\infty = 0.74$ and $\alpha = 0.5$ deg is compared with the solution obtained using the straining of Eq. (10) in Fig. 8. Within the limits of numerical accuracy, it can be seen that the results are identical, giving confidence that the fairly arbitrary choice of straining does not affect the accuracy of the method.

Acknowledgment

The author thanks W.F. Ballhaus and J. Frick for their help in providing the finite-difference solutions used in this paper.

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